

SHARP UPPER BOUND FOR THE FIRST EIGENVALUE

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ABSTRACT. Let M be a closed hypersurface in a noncompact rank-1 symmetric space $(\overline{\mathbb{M}}, ds^2)$ with $-4 \leq K_{\overline{\mathbb{M}}} \leq -1$, or in a complete, simply connected Riemannian manifold \mathbb{M} such that $0 \leq K_{\mathbb{M}} \leq \delta^2$ or $K_{\mathbb{M}} \leq k$ where $k = -\delta^2$ or 0. In this paper we give sharp upperbounds for the first eigenvalue of laplacian of M .

1. INTRODUCTION

Let $(\overline{\mathbb{M}}, g)$ be a complete Riemannian manifold of dimension $n \geq 2$, and M be a closed hypersurface. Starting with the work of Bleecker-Weiner [1], there have been several works which give sharp upper bound for the first eigen value $\lambda_1(M)$ of laplacian on M . In [1], they proved that for a hypersurface M of \mathbb{R}^{n+1} , the first eigenvalue of M is bounded above by $\frac{1}{\text{Vol}(M)} \int_M |A|^2$, where $|A|^2$ is the square of the length of the second fundamental form of M . Reilly [9] proved that if M is a compact n -dimensional manifold which is isometrically immersed in \mathbb{R}^{n+p} , then $\lambda_1(M) \leq \frac{n}{\text{Vol}(M)} \int_M |H|^2$, where H is the mean curvature vector. This result was later extended in various ways to submanifolds of simply connected space forms ([5], [7]). If M is hypersurface in a rank-one symmetric spaces, it was proved in [10] that $\lambda_1(M) \leq \frac{1}{\text{Vol}(M)} \int_M \lambda_1(S(r))$, where $\lambda_1(S(r))$ is the first eigen value of the geodesic sphere in the ambient space with center at the center of mass of M corresponding to the mass distribution function $\frac{1}{t}$. All the above inequalities are sharp as equality holds if and only if the hypersurface M is a geodesic sphere.

For a closed hypersurface M which is contained in a ball of radius less than $\frac{i(\mathbb{M}(k))}{4}$, and bounding a convex domain Ω in the simply connected space form $\mathbb{M}(k)$, $k = 0$ or 1 , the second author proved [11] that

$$\frac{\lambda_1(M)}{\lambda_1(S(R))} \leq \frac{\text{Vol}(M)}{\text{Vol}(S(R))}$$

where R is such that $\text{Vol}(B(R)) = \text{Vol}(\Omega)$. A similar result was also obtained for $k = -1$. Furthermore equality holds if and only if M is a geodesic sphere of radius R .

In this paper we extend the results in [11] to a wider class of Riemannian manifolds. We denote by K_M , the sectional curvature of a Riemannian manifold M . We consider noncompact rank-1 symmetric space $(\overline{\mathbb{M}}, ds^2)$ with the metric ds^2 such that $-4 \leq K_{\overline{\mathbb{M}}} \leq -1$ or complete, simply connected Riemannian manifold \mathbb{M} such that $0 \leq K_{\mathbb{M}} \leq \delta^2$, or $K_{\mathbb{M}} \leq k$, where $k = -\delta^2$ or 0. Let M be a closed hypersurface in \mathbb{M} or $\overline{\mathbb{M}}$. In the case of $0 \leq K_{\mathbb{M}} \leq \delta^2$, we prove the following isoperimetric upper bound

$$\frac{\lambda_1(M)}{\lambda_1(S_\delta(R))} \leq \frac{\text{Vol}(M)}{\text{Vol}(S_\delta(R))}$$

where $S_\delta(R)$ is the geodesic sphere of radius R in the constant curvature space $\mathbb{M}(\delta^2)$ (See theorem 2.3 for the statement). We obtain similar isoperimetric upperbounds in the other cases also (see theorem 2.4 and 2.5 for statements). These upper bounds are sharp and equality holds if and only if the hypersurface is a geodesic sphere.

We refer to [2] and [3] for the basic Riemannian geometry used in this paper.

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2. STATEMENT OF RESULTS

To state the results we need the notion of center of mass of a subset of a Riemannian manifold.

Let (\overline{M}, g) be a $(n+1)$ dimensional complete Riemannian manifold. For a point $p \in \overline{M}$, we denote by $c(p)$ the convexity radius of (\overline{M}, g) at p . For a subset $A \subset B(q, c(q))$, for $q \in \overline{M}$, we let CA denote the convex hull of A . Let $\exp_q : T_q \overline{M} \rightarrow \overline{M}$ be the exponential map and $X = (x_1, x_2, \dots, x_{n+1})$ be the normal coordinate system at q . We identify CA with $\exp_q^{-1}(CA)$ and denote $g_q(X, X)$ as $\|X\|_q^2$ for $X \in T_q \overline{M}$. We state the center of mass theorem below.

Theorem 2.1. *Let A be a measurable subset of (\overline{M}, g) contained in $B(q_0, c(q_0))$ for some point $q_0 \in \overline{M}$. Let $G : [0, 2c(q_0)] \rightarrow \mathbb{R}$ be a continuous function such that G is positive on $(0, 2c(q_0))$. Then there exists a point $p \in CA$ such that*

$$\int_A G(\|X\|_p) X dV = 0,$$

where $X = (x_1, x_2, \dots, x_{n+1})$ is a geodesic normal coordinate system at p .

For a proof see [10] or [7].

Definition 2.2. The point p in the above theorem is called as a center of mass of the measurable subset A with respect to the mass distribution function G .

Before stating the results, we fix some notations which will be used throughout the paper. Let (\overline{M}, ds^2) be a noncompact rank-1 symmetric space with the metric ds^2 such that the sectional curvature satisfies $-4 \leq K_{\overline{M}} \leq -1$. Let the dimension of (\overline{M}, ds^2) be kn , where $k = \dim_{\mathbb{R}} \mathbb{K}$; $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{C}a$. Fix a point $p \in \overline{M}$ and let γ be a geodesic starting at p . Then the volume density function along γ at the point $\gamma(r)$ is given by $\sinh^{kn-1} r \cosh^{k-1} r$. Also we denote by $S(r)$, the geodesic sphere of radius r with center p and by $\Delta_{S(r)}$, the Laplacian of $S(r)$. Let $\lambda_1(S(r))$ be the first eigenvalue of $\Delta_{S(r)}$. It is well known ([7], [10]) that for $r > 0$

$$\lambda_1(S(r)) = \frac{kn-1}{\sinh^2 r} - \frac{k-1}{\cosh^2 r}.$$

For a given $\delta > 0$, let \mathbb{M} denote a complete, simply connected Riemannian manifold of dimension $(n+1)$ such that the sectional curvature satisfies one of the following conditions:

- (1) $0 \leq K_{\mathbb{M}} \leq \delta^2$
- (2) $K_{\mathbb{M}} \leq 0$
- (3) $K_{\mathbb{M}} \leq -\delta^2$.

Also we denote by $\mathbb{M}(k)$, the simply connected space form of dimension $(n+1)$ with constant curvature $k = \pm\delta^2$ or 0. For $r \geq 0$, let

$$\sin_{\delta} r = \begin{cases} \frac{1}{\delta} \sin \delta r & \text{if } 0 \leq K \leq \delta^2 \\ r & \text{if } K \leq 0 \\ \frac{1}{\delta} \sinh \delta r & \text{if } K \leq -\delta^2 \end{cases} \quad \text{and} \quad \cos_{\delta} r = \begin{cases} \cos \delta r & \text{if } 0 \leq K \leq \delta^2 \\ 1 & \text{if } K \leq 0 \\ \cosh \delta r & \text{if } K \leq -\delta^2. \end{cases}$$

Let M be a closed hypersurface in \mathbb{M} or in $\overline{\mathbb{M}}$ and Ω be a bounded domain whose boundary is M . In the case of \mathbb{M} with $0 \leq K_{\mathbb{M}} \leq \delta^2$, we always assume that M is contained in a ball of radius less than $\min(\frac{\pi}{4\delta}, \text{inj}(\mathbb{M}))$. Let $p \in C\Omega$ be a center of mass corresponding to the function $\frac{\sin_{\delta} r}{r}$. The geodesic polar coordinate system centered at p is denoted by (r, u) where $r > 0$ and $u \in U_p \mathbb{M}$ (or $U_p \overline{\mathbb{M}}$). For any $q \in M$, let γ_q be the unique unit speed geodesic segment joining p and q with $\gamma_q'(0) = u$. We write $d(p, q)$ as $t_q(u)$. Consider $W \subset T_p \mathbb{M}$ such that $\Omega = \exp_p(W)$. Fix a point $p_0 \in \mathbb{M}(k)$ and an isometry $i : T_p \mathbb{M} \rightarrow T_{p_0} \mathbb{M}(k)$. Let $\Omega_{\delta} = \exp_{p_0}(i(W))$, $M_{\delta} = \partial \Omega_{\delta}$ and for $\bar{q} \in M_{\delta}$ we write $d(p_0, \bar{q}) = t_{\bar{q}}(\bar{u})$ where \bar{u} is the tangent at p_0 of the unit speed geodesic segment $\gamma_{\bar{q}}$ joining between p_0 and \bar{q} . We also denote by ϕ and ϕ_{δ} , the volume density functions of \mathbb{M} and $\mathbb{M}(k)$ along the radial geodesics starting at p and p_0 respectively.

With these notations we state the main results.

Theorem 2.3. *Let \mathbb{M} be a complete, simply connected $(n+1)$ dimensional manifold such that $0 \leq K_{\mathbb{M}} \leq \delta^2$ or $K_{\mathbb{M}} \leq 0$. Let M be a closed hypersurface in \mathbb{M} which encloses a bounded region Ω . Then*

$$\frac{\lambda_1(M)}{\lambda_1(S_\delta(R))} \leq \frac{\text{Vol}(M)}{\text{Vol}(S_\delta(R))}$$

where $R > 0$ is such that $\text{Vol}(\Omega_\delta) = \text{Vol}(B_\delta(R))$; here $B_\delta(R)$ and $S_\delta(R)$ are the geodesic ball and geodesic sphere respectively of radius R in the constant curvature space $\mathbb{M}(k)$, where $k = \delta^2$ or 0 .

Further, the equality holds if and only if M is a geodesic sphere in \mathbb{M} and Ω is isometric to $B_\delta(R)$.

For the case $K \leq -\delta^2$, we have

Theorem 2.4. *Let \mathbb{M} be a complete, simply connected $(n+1)$ dimensional manifold such that $K \leq -\delta^2$. Let M be a closed hypersurface in \mathbb{M} which encloses the bounded region Ω . Then*

$$\frac{\lambda_1(M)}{\lambda_1(S_\delta(R))} \leq \frac{\text{Vol}(M)}{\text{Vol}(S_\delta(R))} + \frac{1}{n \text{Vol}(S_\delta(R))} \int_M \|\nabla^M \sin_\delta r\|^2$$

where $R > 0$ is such that $\text{Vol}(\Omega_\delta) = \text{Vol}(B_\delta(R))$; here $B_\delta(R)$ and $S_\delta(R)$ are the geodesic ball and geodesic sphere respectively of radius R in the constant curvature space $\mathbb{M}(-\delta^2)$.

Further, the equality holds if and only if M is a geodesic sphere and Ω is isometric to $B_\delta(R)$.

In the case of rank-1 symmetric space of noncompact type, we have

Theorem 2.5. *Let $(\overline{\mathbb{M}}, ds^2)$ be a non compact rank-1 symmetric space and M be a closed hypersurface in \mathbb{M} which encloses the bounded region Ω . Then*

$$\lambda_1(M) \leq \lambda_1(S(R)) \left(\frac{\text{Vol}(M)}{\text{Vol}(S(R))} \right) + \frac{1}{\sinh^2 R \text{Vol}(S(R))} \int_M \|\nabla^M \sinh r\|^2$$

where $R > 0$ is such that $\text{Vol}(\Omega) = \text{Vol}(B(R))$; here $B(R)$ and $S(R)$ are the geodesic ball and geodesic sphere respectively of radius R .

Further, the equality holds if and only if M is a geodesic sphere of radius R .

3. PRELIMINARIES

Let M be a closed hypersurface in \mathbb{M} or in $\overline{\mathbb{M}}$ and let Ω be the bounded domain whose boundary is M . Fix a point $p \in \Omega$. Then for every point $q \in M$, there exist unique geodesic segment γ_q such that $\gamma_q(0) = p, \gamma'_q(0) = u$ and $\gamma_q(t_q(u)) = q$. We observe that this geodesic segment may intersect M at points other than q . For $u \in U_p \mathbb{M}$ (or $U_p \overline{\mathbb{M}}$), let

$$r(u) = \max\{r > 0 \mid \exp_p(ru) \in M\}$$

and define

$$A = \{\exp_p(r(u)u) \mid u \in U_p \mathbb{M} \text{ (or } U_p \overline{\mathbb{M}})\}.$$

Then $A \subset M$ and hence for any non-negative measurable function f on M , we have $\int_M f \geq \int_A f$.

We now prove the following lemma which is crucial in the proofs of main results.

Lemma 3.1. *Let M be a closed hypersurface in \mathbb{M} or in $\overline{\mathbb{M}}$ and Ω be a bounded domain with boundary $\partial\Omega = M$. Fix a point $p \in \Omega$. Then the following holds:*

(1) *If $M \subseteq \mathbb{M}$, then*

$$\int_M \sin_\delta^2 d(p, q) dm \geq \text{Vol}(S_\delta(p_0, R)) \sin_\delta^2 R$$

where dm is the measure on M , $S_\delta(p_0, R)$ is the geodesic sphere and $B_\delta(p_0, R)$ is the geodesic ball of radius R , centered at p_0 in the space form $\mathbb{M}(k)$ and $R > 0$ is such that $\text{Vol}(\Omega_\delta) = \text{Vol}(B_\delta(p_0, R))$.

Further, the equality holds if and only if M is a geodesic sphere in \mathbb{M} and Ω is isometric to $B_\delta(p_0, R)$.

(2) If $M \subseteq \overline{\mathbb{M}}$, then

$$\int_M \sinh^2 d(p, q) dm \geq \text{Vol}(S(p, R)) \sinh^2 R$$

where dm is the measure on M , $S(p, R)$ is the geodesic sphere and $B(p, R)$ is the geodesic ball of radius R centered at p in $\overline{\mathbb{M}}$ and $R > 0$ is such that $\text{Vol}(\Omega) = \text{Vol}(B(p, R))$.

The equality holds if and only if M is a geodesic sphere centered at p of radius R .

Proof. We begin by considering $M \subset \mathbb{M}$. Let $q \in M$ and $\phi(t_q(u))$ be the volume density of the geodesic sphere $S(p, t_q(u))$ at the point q . Let $\theta(q)$ be the angle between the unit outward normal $\eta(q)$ to M and the radial vector $\partial r(q)$. Let du be the spherical volume density of the unit sphere $U_p \mathbb{M}$. Then we know that ([8], p.385, or [12], p.1097) $dm(q) = \sec \theta(q) \phi(t_q(u)) du$. Hence,

$$\begin{aligned} \int_M \sin_\delta^2 d(p, q) dm(q) &\geq \int_A \sin_\delta^2 d(p, q) dm(q) \\ &= \int_{U_p \mathbb{M}} \sin_\delta^2 t_q(u) \sec \theta(q) \phi(t_q(u)) du \\ &\geq \int_{U_p \mathbb{M}} \sin_\delta^2 t_q(u) \phi(t_q(u)) du. \end{aligned}$$

For $q \in M$, consider the unit speed geodesic segment γ_q in \mathbb{M} joining p and q and the corresponding geodesic segment $\gamma_{\bar{q}}$ joining p_0 and $\bar{q} \in M_\delta$ in the space form $\mathbb{M}(k)$. Then by Rauch comparison theorem [3] it follows that $l(\gamma_q) \geq l(\gamma_{\bar{q}})$ and hence $t_q(u) \geq t_{\bar{q}}(\bar{u})$. By Gunther's volume comparison theorem [6] we also have $\phi(t_q(u)) \geq \phi_\delta(t_q(u)) = \sin_\delta^n t_q(u)$ along the geodesics γ_p and $\gamma_{\bar{q}}$ respectively. Hence,

$$\begin{aligned} \int_M \sin_\delta^2 d(p, q) dm(q) &\geq \int_{U_p \mathbb{M}} \sin_\delta^2 t_q(u) \phi(t_q(u)) du \\ &\geq \int_{U_p \mathbb{M}} \sin_\delta^2 t_q(u) \phi_\delta(t_q(u)) du \\ &\geq \int_{U_p \mathbb{M}(k)} \sin_\delta^2 t_{\bar{q}}(\bar{u}) \phi_\delta(t_{\bar{q}}(\bar{u})) d\bar{u} \\ &= \int_{U_p \mathbb{M}(k)} \sin_\delta^{n+2} t_{\bar{q}}(\bar{u}) d\bar{u} \\ &= (n+2) \int_{U_p \mathbb{M}(k)} \int_0^{t_{\bar{q}}(\bar{u})} \sin_\delta^{n+1} r \cos_\delta r dr d\bar{u} \\ &\geq (n+2) \int_{\Omega_\delta} \sin_\delta r \cos_\delta r dV. \end{aligned}$$

Let $R > 0$ be such that $\text{Vol}(\Omega_\delta) = \text{Vol}(B_\delta(p_0, R))$ and $f(r) = \sin_\delta r \cos_\delta r$. We observe that

$$f \text{ is increasing on } \begin{cases} [0, \frac{\pi}{4\delta}) & \text{if } 0 \leq K \leq \delta^2 \\ [0, \infty) & \text{if } K \leq 0 \end{cases}$$

and $\text{Vol}(\Omega_\delta \setminus (\Omega_\delta \cap B_\delta(p_0, R))) = \text{Vol}(B_\delta(p_0, R) \setminus (\Omega_\delta \cap B_\delta(p_0, R)))$. Using these facts we get

$$\begin{aligned} \int_{\Omega_\delta} f(r) dV &= \int_{\Omega_\delta \cap B_\delta(p_0, R)} f(r) dV + \int_{\Omega_\delta \setminus (\Omega_\delta \cap B_\delta(p_0, R))} f(r) dV \\ &= \int_{B_\delta(p_0, R)} f(r) dV - \int_{B_\delta(p_0, R) \setminus \Omega_\delta \cap B_\delta(p_0, R)} f(r) dV \\ &\quad + \int_{\Omega_\delta \setminus (\Omega_\delta \cap B_\delta(p_0, R))} f(r) dV \end{aligned}$$

$$\begin{aligned}
&\geq \int_{B_\delta(p_0, R)} f(r) dV - \int_{B_\delta(p_0, R) \setminus \Omega_\delta \cap B_\delta(p_0, R)} f(r) dV \\
&\quad + \int_{\Omega_\delta \setminus (\Omega_\delta \cap B_\delta(p_0, R))} f(R) dV \\
&= \int_{B_\delta(p_0, R)} f(r) dV + \int_{B_\delta(p_0, R) \setminus \Omega_\delta \cap B_\delta(p_0, R)} (f(R) - f(r)) dV \\
&\geq \int_{B_\delta(p_0, R)} f(r) dV \\
&= \int_{U_p \mathbb{M}(k)} \int_0^R f(r) \sin_\delta^n r dr d\bar{u} \\
&= \int_{U_p \mathbb{M}(k)} \int_0^R \sin_\delta^{n+1} r \cos_\delta r dr d\bar{u} \\
&= \frac{1}{n+2} \int_{U_p \mathbb{M}(k)} \sin_\delta^{n+2} R d\bar{u} \\
&= Vol(S_\delta(R)) \frac{\sin_\delta^2 R}{n+2}.
\end{aligned}$$

Thus we get

$$\int_M \sin_\delta^2 d(p, q) dm(q) \geq Vol(S_\delta(R)) \sin_\delta^2 R.$$

Further the equality holds in the above inequality if and only if the following conditions hold:

- (1) $\sec \theta(q) = 1$ and $l(\gamma_q) = l(\gamma_{\bar{q}})$ for all points $q \in M$.
- (2) $\phi(r) = \phi_\delta(r)$ for $r \leq \text{diam}(\mathbb{M})$ along the geodesics γ_p and $\gamma_{\bar{q}}$ respectively.
- (3) $Vol(B_\delta(p_0, R) \setminus (\Omega_\delta \cap B_\delta(p_0, R))) = 0$.

Now $\sec \theta(q) = 1$ implies that the outward normal $\eta(q) = \partial r(q)$. Thus the first condition implies that $\eta(q) = \partial r(q)$ for all points in $q \in M$. This shows that M is a geodesic sphere centered at p . The equality criteria in Gunther's volume comparison theorem ([4], [6]) says that if $\phi(r) = \phi_\delta(r)$ for $r \leq R \leq \text{diam}(\mathbb{M})$ then the geodesic balls $B(p, R)$ and $B_\delta(p_0, R)$ are isometric. Hence we see that Ω is isometric to $B_\delta(p_0, R)$.

Now suppose $M \subset \overline{\mathbb{M}}$. For the noncompact rank-1 symmetric spaces $(\overline{\mathbb{M}}, ds^2)$, the density function ϕ along the geodesics starting at the point p is given by $\phi(r) = \sinh^{kn-1} r \cosh^{k-1} r$. The computation is slightly different in this case. As an illustration we give below an outline of the proof for \mathbb{CH}^n . For other noncompact rank-1 symmetric spaces the proof follows similarly.

We proceed as in the earlier computation to get,

$$\begin{aligned}
\int_M \sinh^2 d(p, q) dm(q) &\geq \int_A \sinh^2 d(p, q) dm(q) \\
&= \int_{U_p \mathbb{M}} \sinh^2 t_q(u) \sec \theta(q) \phi(t_q(u)) du \\
&\geq \int_{U_p \mathbb{M}} \sinh^2 t_q(u) \phi(t_q(u)) du \\
&= \int_{U_p \mathbb{M}} \sinh^{2n+1} t_q(u) \cosh t_q(u) du \\
&= \int_{U_p \mathbb{M}} \int_0^{t_q(u)} f(r) \sinh^{2n-1} r \cosh r dr du \\
&\geq \int_\Omega f(r) dV
\end{aligned}$$

where $f(r) = (2n+1) \sinh r \cosh r + \sinh^2 r \tanh r$. Notice that $f(r)$ is increasing for $r \geq 0$. The rest of the proof follows the same way as in the earlier case. \square

A computation similar to the above lemma proves the following lemma.

Lemma 3.2. *Let $(\overline{\mathbb{M}}, ds^2)$ be a noncompact rank-1 symmetric space. Let M be a closed hypersurface in $\overline{\mathbb{M}}$ and Ω be the bounded domain with boundary $\partial\Omega = \mathbb{M}$. Fix a point $p \in \Omega$. Then*

$$\int_M \tanh^2 d(p, q) dm \geq \text{Vol}(S(p, R)) \tanh^2 R$$

where $S(p, R)$ is the geodesic sphere and $B(p, R)$ is the geodesic ball of radius R centered at p in $\overline{\mathbb{M}}$ and $R > 0$ is such that $\text{Vol}(\Omega) = \text{Vol}(B(p, R))$.

The equality holds if and only if M is a geodesic sphere centered at p of radius R .

Remark 3.3. Lemma 3.1 and lemma 3.2 are also valid for hypersurfaces in compact rank-1 symmetric spaces.

4. PROOF OF RESULTS

Let M be a closed hypersurface of \mathbb{M} and $p \in \mathbb{M}$ be a center of mass corresponding to the mass distribution function $\frac{\sin_\delta r}{r}$. Let $X = (x_1, x_2, \dots, x_{n+1})$ be the geodesic normal coordinate system centered at p . Consider the functions $g_i = f \cdot \frac{x_i}{r}$ where $f = \sin_\delta r$. Then $\int_M g_i = 0$ for $1 \leq i \leq n+1$. Using g_i 's as test functions in the Rayleigh quotient, we have

$$\begin{aligned} (4.1) \quad \lambda_1(M) \int_M \sum_{i=1}^{n+1} g_i^2 dm &\leq \int_M \sum_{i=1}^{n+1} \|\nabla^M g_i\|^2 dm \\ &= \int_M \sum_{i=1}^{n+1} g_i \Delta_M g_i dm \\ &= \int_M f \Delta_M f dm + \int_M f^2 \sum_{i=1}^{n+1} f_i \Delta_M f_i dm \end{aligned}$$

where $f_i = \frac{x_i}{r}$. But $\sum_{i=1}^{n+1} g_i^2 = f^2$. Thus we get

$$(4.2) \quad \lambda_1(M) \int_M f^2 dm \leq \int_M f \Delta_M f dm + \int_M f^2 \sum_{i=1}^{n+1} f_i \Delta_M f_i dm.$$

We now decompose the laplacian on M as

$$\Delta_M = \frac{\partial^2}{\partial \eta^2} + \text{Tr}(A) \frac{\partial}{\partial \eta} + \Delta$$

where η is the unit outward normal to M , A is the Weingarten map of M and Δ is the laplacian on \mathbb{M} . We do another decomposition of Δ along the radial geodesic starting from p as

$$\Delta = -\frac{\partial^2}{\partial r^2} - \text{Tr}(A) \frac{\partial}{\partial r} + \Delta_{S(r)}$$

where $\Delta_{S(r)}$ is the laplacian of $S(r)$. We also notice that $\sum_{i=1}^{n+1} f_i \frac{\partial f_i}{\partial \eta} = \frac{1}{2} \sum_{i=1}^{n+1} \langle \nabla f_i^2, \eta \rangle = 0$ and $\frac{\partial}{\partial r}(f_i) = 0$. Using these we have

$$\begin{aligned}
\sum_{i=1}^{n+1} f_i \Delta_M f_i &= \sum_{i=1}^{n+1} f_i \Delta f_i + \sum_{i=1}^{n+1} f_i \frac{\partial^2 f_i}{\partial \eta^2} \\
&= \sum_{i=1}^{n+1} f_i \Delta_{S_r} f_i - \sum_{i=1}^{n+1} \left(\frac{\partial f_i}{\partial \eta} \right)^2 \\
&= \sum_{i=1}^{n+1} \|\nabla^{S_r} f_i\|^2 - \sum_{i=1}^{n+1} \left(\frac{\partial f_i}{\partial \eta} \right)^2 \\
(4.3) \quad &= \frac{1}{r^2} \sum_{i=1}^{n+1} \|\nabla^{S_r} x_i\|^2 - \sum_{i=1}^{n+1} \left(\frac{\partial f_i}{\partial \eta} \right)^2.
\end{aligned}$$

Thus (4.2) becomes

$$\begin{aligned}
\lambda_1(M) \int_M f^2 dm &\leq \int_M \|\nabla^M f\|^2 dm + \sum_{i=1}^{n+1} \int_M \frac{f^2 \|\nabla^{S_r} x_i\|^2}{r^2} dm \\
(4.4) \quad &\quad - \sum_{i=1}^{n+1} \int_M f^2 \left(\frac{\partial f_i}{\partial \eta} \right)^2 dm.
\end{aligned}$$

Now let M be a closed hypersurface in the noncompact rank-1 symmetric space $\overline{\mathbb{M}}$. Let $p \in \overline{\mathbb{M}}$ be a center of mass corresponding to the mass distribution function $\frac{\sinh r}{r}$ and let $f = \sinh r$. The same calculation as above shows that (4.4) holds in this case also.

We now estimate $\sum \|\nabla^{S_r} x_i\|^2$.

First, consider the case $M \subset \overline{\mathbb{M}}$. It is known [10] that $\Delta_{S(r)} f_i = \lambda_1(S(r)) f_i$, where $\lambda_1(S(r))$ is the first eigen value of the laplacian $\Delta_{S(r)}$ of the geodesic sphere $S(r) \subset \overline{\mathbb{M}}$. Thus

$$\sum_{i=1}^{kn} \frac{\|\nabla^{S_r} x_i\|^2}{r^2} = \sum_{i=1}^{kn} f_i \Delta_{S_r} f_i = \lambda_1(S(r)).$$

Hence (4.3) can be written as

$$\sum_{i=1}^{kn} f_i \Delta_M f_i = \lambda_1(S(r)) - \sum_{i=1}^{kn} \left(\frac{\partial f_i}{\partial \eta} \right)^2.$$

Substituting this in (4.4) we get for a closed hypersurface M in the noncompact rank-1 symmetric space $\overline{\mathbb{M}}$,

$$(4.5) \quad \lambda_1(M) \int_M f^2 dm \leq \int_M \|\nabla^M f\|^2 dm + \int_M f^2 \left(\lambda_1(S(r)) - \sum_{i=1}^{kn} \left(\frac{\partial f_i}{\partial \eta} \right)^2 \right) dm.$$

The following lemma gives an estimate of $\sum \|\nabla^M x_i\|^2$ in the general case.

Lemma 4.1. *Let $\delta > 0$ be given, and \mathbb{M} be a complete, simply connected Riemannian manifold of dimension $(n+1)$ such that the sectional curvature satisfies $K_{\mathbb{M}} \leq k$ where $k = \pm\delta^2$ or 0. Let M be a closed hypersurface and Ω be a bounded domain such that $\partial\Omega = M$. Fix a point $p \in \Omega$ and let $X = (x_1, x_2, \dots, x_{n+1})$ be the geodesic normal coordinate system at p . For $K_{\mathbb{M}} \leq \delta^2$, we assume that Ω is contained in a geodesic sphere of radius $R < \frac{\pi}{\delta}$. Then for every point $q \in M$,*

$$\sum_{i=1}^{n+1} \|\nabla^M x_i(q)\|^2 \leq \frac{n d(p, q)^2}{\sin^2 \delta d(p, q)}.$$

Proof. Let $q \in M$ and (e_1, \dots, e_n) be an orthonormal basis of $T_q M$. Then

$$\sum_{i=1}^{n+1} \|\nabla^M x_i\|^2 = \sum_{i=1}^{n+1} \sum_{l=1}^n \langle \nabla^M x_i, e_l \rangle^2 = \sum_{i=1}^{n+1} \sum_{l=1}^n \langle \nabla x_i, e_l \rangle^2.$$

Let $e_l = d(\exp_p) \bar{e}_l$. Note that $\langle \nabla x_i, e_l \rangle = e_l(x_i) = \bar{e}_l(x_i \circ \exp_p)$ is the i -th component of \bar{e}_l in the geodesic normal coordinate at p . Thus

$$\sum_{i=1}^{n+1} \langle \nabla x_i, e_l \rangle^2 = \|\bar{e}_l\|^2.$$

Consider a unit speed geodesic γ in \mathbb{M} such that $\gamma(0) = p$ and $\gamma(r) = q$. Let J_l be the Jacobi field along γ such that $J_l(0) = 0$ and $J'_l(0) = \bar{e}_l$. Then $e_l = d(\exp_p) \bar{e}_l = \frac{1}{r} J_l(r)$. Let $\mathbb{M}(k)$ be the space form with constant curvature $k = \pm \delta^2$ or 0 . Fix a point $p_0 \in \mathbb{M}(k)$ and a unit speed geodesic $\bar{\gamma}$ such that $\bar{\gamma}(0) = p_0$. Let \bar{u} be a unit vector at p_0 and $E(t)$ be the vector field obtained by parallel translating \bar{u} along $\bar{\gamma}(t)$. Consider the Jacobi field

$$J_\delta(t) = \sin_\delta t \parallel J'_l(0) \parallel E(t)$$

along $\bar{\gamma}$. By the Rauch comparison theorem

$$\parallel J_\delta(t) \parallel \leq \parallel J_l(t) \parallel \quad \text{for} \quad \begin{cases} 0 \leq t < \frac{\pi}{\delta} & \text{if } K \leq \delta^2 \\ t \geq 0 & \text{if } K \leq -\delta^2 \text{ or } 0. \end{cases}$$

Hence $\parallel e_l \parallel^2 = \frac{1}{r^2} \parallel J_l(r) \parallel^2 \geq \frac{1}{r^2} \parallel J_\delta(r) \parallel^2 = \frac{\sin_\delta^2 r}{r^2} \parallel J'_l(0) \parallel^2$, which implies

$$\parallel \bar{e}_l \parallel^2 = \parallel J'_l(0) \parallel^2 \leq \frac{r^2}{\sin_\delta^2 r}.$$

Thus we get

$$\sum_{i=1}^{n+1} \|\nabla^M x_i(q)\|^2 = \sum_{i=1}^{n+1} \sum_{l=1}^n \langle \nabla x_i, e_l \rangle^2 = \sum_{l=1}^n \|\bar{e}_l\|^2 \leq \frac{n d(p, q)^2}{\sin_\delta^2 d(p, q)}.$$

□

Proof of theorem 2.3. We first consider the case $0 \leq K_{\mathbb{M}} \leq \delta^2$. Recall the inequality (4.4)

$$\begin{aligned} \lambda_1(M) \int_M f^2 dm &\leq \int_M \|\nabla^M f\|^2 dm + \int_M \frac{f^2}{r^2} \sum_{i=1}^{n+1} \|\nabla^{S_r} x_i\|^2 dm \\ &\quad - \int_M f^2 \sum_{i=1}^{n+1} \left(\frac{\partial f_i}{\partial \eta} \right)^2 dm. \end{aligned}$$

Applying lemma 4.1 for the geodesic sphere S_r , we get $\sum_{i=1}^{n+1} \|\nabla^{S_r} x_i\|^2 \leq \frac{n r^2}{\sin_\delta^2 r}$. Substituting this in the above inequality, we get

$$\begin{aligned} \lambda_1(M) \int_M \sin_\delta^2 d(p, q) dm(q) &\leq n \text{Vol}(M) + \int_M \|\nabla^M \sin_\delta d(p, q)\|^2 dm(q) \\ (4.6) \quad &\quad - \int_M \sin_\delta^2 d(p, q) \sum_{i=1}^{n+1} \left(\frac{\partial f_i}{\partial \eta}(q) \right)^2 dm(q). \end{aligned}$$

We now estimate $\sum_{i=1}^{n+1} \left(\frac{\partial f_i}{\partial \eta}(q) \right)^2$ for $q \in M$.

For $q \in M$, let $\eta(q) = a \partial r(q) + b v$ where $v \in T_q \mathbb{M}$, $\parallel v \parallel = 1$, $\langle \partial r(q), v \rangle = 0$. Note that $b^2 = \|\nabla^M r(q)\|^2$ and $\langle \nabla f_i, \eta \rangle(q) = a \langle \nabla f_i, \partial r \rangle(q) + b \langle \nabla f_i(q), v \rangle$. But $\nabla f_i = \frac{\nabla x_i}{r} - \frac{x_i}{r^2} \partial r$. Thus $\langle \nabla f_i, \eta \rangle(q) = \frac{b}{r} \langle \nabla x_i(q), v \rangle$. Let $v = d(\exp_p) \bar{v} = \frac{1}{r} J(r)$ where J is the Jacobi field along the unit speed geodesic γ_q

which joins p and q such that $J(0) = 0$ and $J'(0) = \bar{v}$. Note that $\langle \nabla x_i(q), v \rangle = v(x_i) = \bar{v}(x_i \circ \exp_p)$ is the i -th component of \bar{v} in the geodesic normal coordinate at p . Thus

$$\sum_{i=1}^{n+1} \langle \nabla x_i(q), v \rangle^2 = \|\bar{v}\|^2.$$

Consider \mathbb{R}^{n+1} and fix a point $p_0 \in \mathbb{R}^{n+1}$. Let α be a unit speed geodesic such that $\alpha(0) = p_0$. Let \bar{u} be a unit vector at p_0 and $W(t)$ be the vector field obtained by parallel translating \bar{u} along $\alpha(t)$. Consider the Jacobi field along $J_0(t)$ along $\alpha(t)$ given by

$$J_0(t) = \|J'(0)\| tW(t).$$

By Rauch comparison theorem we see that $\|J(t)\| \leq \|J_0(t)\|$ for $t < \text{inj}(\mathbb{M})$. Thus

$$1 = \|v\|^2 = \frac{1}{r^2} \|J(r)\|^2 \leq \frac{1}{r^2} \|J'(0)\|^2 r^2 = \|\bar{v}\|^2$$

which gives

$$\sum_{i=1}^{n+1} \langle \nabla f_i(q), \eta(q) \rangle^2 = \frac{b^2}{r^2} \sum_{i=1}^{n+1} \langle \nabla x_i(q), v \rangle^2 = \frac{b^2}{r^2} \|\bar{v}\|^2 \geq \frac{b^2}{r^2}.$$

Substituting this, (4.6) becomes

$$\begin{aligned} \lambda_1(M) \int_M \sin_\delta^2 d(p, q) dm(q) &\leq n \text{Vol}(M) + \int_M \|\nabla^M \sin_\delta d(p, q)\|^2 dm(q) \\ &\quad - \int_M \sin_\delta^2 d(p, q) \frac{\|\nabla^M r\|^2}{r^2} dm(q). \end{aligned}$$

Also we have $\|\nabla^M \sin_\delta r\|^2 = \|\nabla^M r\|^2 \cos_\delta^2 r$. Hence

$$\lambda_1(M) \int_M \sin_\delta^2 d(p, q) dm \leq n \text{Vol}(M) + \int_M \|\nabla^M r\|^2 \left(\cos_\delta^2 r - \frac{\sin_\delta^2 r}{r^2} \right) dm(q).$$

As $\tan \delta r = \frac{\sin \delta r}{\cos \delta r} \geq \delta r$, for $0 \leq r < \frac{\pi}{\delta}$, we get

$$\lambda_1(M) \int_M \sin_\delta^2 d(p, q) dm \leq n \text{Vol}(M).$$

Now by lemma 3.1, $\int_M \sin_\delta^2 d(p, q) dm \geq \text{Vol}(S_\delta(R)) \sin_\delta^2 R$. Substituting this in the above inequality we get,

$$\lambda_1(M) \leq \frac{n}{\sin_\delta^2 R} \left(\frac{\text{Vol}(M)}{\text{Vol}(S_\delta(R))} \right).$$

Since $\frac{n}{\sin_\delta^2 R} = \lambda_1(S_\delta(R))$, we have

$$(4.7) \quad \lambda_1(M) \leq \lambda_1(S_\delta(R)) \left(\frac{\text{Vol}(M)}{\text{Vol}(S_\delta(R))} \right)$$

which is the desired inequality.

As $\tan \delta r > \delta r$ for $0 < r < \frac{\pi}{\delta}$, the equality holds if and only if $\|\nabla^M r\| = 0$ and the equality in lemma(3.1) holds. This shows that equality in (4.7) holds if and only if M is geodesic sphere in \mathbb{M} and Ω is isometric to $B_\delta(R)$.

Next consider the case $K_{\mathbb{M}} \leq 0$. Notice that $g_i = x_i$ for $i = 1, \dots, n+1$. Hence (4.1) can be written as

$$(4.8) \quad \lambda_1(M) \int_M \sum_{i=1}^{n+1} x_i^2 dm \leq \int_M \sum_{i=1}^{n+1} \|\nabla^M x_i\|^2 dm.$$

By lemma 4.1 we get

$$\sum_{i=1}^{n+1} \|\nabla^M x_i\|^2 \leq n.$$

Also by lemma 3.1, we have

$$\int_M \sum_{i=1}^{n+1} x_i^2 dm = \int_M r^2 dm \geq R^2 \text{Vol}(S(R)).$$

Substituting these into (4.8) and using the fact that $\frac{n}{R^2} = \lambda_1(S(R))$, we get the desired inequality

$$\lambda_1(M) \leq \lambda_1(S(R)) \left(\frac{\text{Vol}(M)}{\text{Vol}(S(R))} \right).$$

Further, the equality condition follows from the equality condition in the lemma 3.1. \square

Proof of theorem 2.4. The proof in this case is similar to the proof of theorem 2.3 except that we do not have a similar estimate of

$$\int_M \|\nabla^M \sin_\delta d(p, q)\|^2 dm(q) - \int_M \sin_\delta^2 d(p, q) \sum_{i=1}^{n+1} \left(\frac{\partial f_i}{\partial \eta}(q) \right)^2 dm(q).$$

Thus from (4.6),

$$\lambda_1(M) \int_M \sin_\delta^2 d(p, q) dm(q) \leq n \text{Vol}(M) + \int_M \|\nabla^M \sin_\delta d(p, q)\|^2 dm(q).$$

By lemma 3.1, we get the desired inequality

$$(4.9) \quad \frac{\lambda_1(M)}{\lambda_1(S_\delta(R))} \leq \frac{\text{Vol}(M)}{\text{Vol}(S_\delta(R))} + \frac{1}{n \text{Vol}(S_\delta(R))} \int_M \|\nabla^M \sin_\delta r\|^2.$$

The equality holds if and only if the equality in lemma 3.1 holds and $\frac{\partial f_i}{\partial \eta}(q) = 0$ for all $i = 1, \dots, n+1$ and for all points $q \in M$. The later is true if and only if $\eta(q) = \partial r(q)$ for all points $q \in M$, which implies that M is a geodesic sphere. Thus the equality in (4.9) holds if and only if M is a geodesic sphere and Ω is isometric to $B_\delta(R)$. \square

Proof of theorem 2.5. We recall the inequality (4.5),

$$\begin{aligned} \lambda_1(M) \int_M \sinh^2 r dm &\leq \int_M \|\nabla^M \sinh r\|^2 dm + \int_M \sinh^2 r \lambda_1(S(r)) dm \\ &\quad - \int_M \sum_{i=1}^{kn} \left(\frac{\partial f_i}{\partial \eta} \right)^2 dm. \end{aligned}$$

Substituting for $\lambda_1(S(r))$ in the above inequality we get

$$\begin{aligned} \lambda_1(M) \int_M f^2 dm &\leq (kn - 1) \text{Vol}(M) - (k - 1) \int_M \tanh^2 r dm \\ &\quad + \int_M \|\nabla^M \sinh r\|^2 dm. \end{aligned}$$

By lemma 3.1 and lemma 3.2, we get

$$\begin{aligned} \lambda_1(M) \text{Vol}(S(R)) \sinh^2 R &\leq ((kn - 1) - (k - 1) \tanh^2 R) \text{Vol}(M) \\ &\quad + \int_M \|\nabla^M \sinh r\|^2 dm. \end{aligned}$$

This implies

$$\begin{aligned} \lambda_1(M) &\leq \left(\frac{(kn - 1)}{\sinh^2 R} - \frac{(k - 1)}{\cosh^2 R} \right) \frac{\text{Vol}(M)}{\text{Vol}(S(R))} \\ &\quad + \frac{1}{\sinh^2 R \text{Vol}(S(R))} \int_M \|\nabla^M \sinh r\|^2 dm. \end{aligned}$$

Using the fact that $\lambda_1(S(R)) = \frac{(kn-1)}{\sinh^2 R} - \frac{(k-1)}{\cosh^2 R}$, we get the desired inequality

$$(4.10) \quad \lambda_1(M) \leq \lambda_1(S(R)) \left(\frac{\text{Vol}(M)}{\text{Vol}(S(R))} \right) + \frac{1}{\sinh^2 R \text{Vol}(S(R))} \int_M \|\nabla^M \sinh r\|^2 dm.$$

The equality condition follows from the equality criterion in lemma 3.1 and lemma 3.2 and the fact that $\frac{\partial f_i}{\partial \eta}(q) = 0$ for all $i = 1, \dots, kn$ and for all points $q \in M$ happens if and only if M is a geodesic sphere. \square

Remark 4.2. The same proof fails for compact rank-1 symmetric spaces. In this case the right hand side of (4.10) will have the following term

$$\frac{1}{\sinh^2 R \text{Vol}(S(R))} \int_M (\|\nabla^M \sinh r\|^2 + (\tan^2 r - \tan^2 R)) dm$$

and we do not have an estimate for the error term $\int_M (\tan^2 r - \tan^2 R)$.

Remark 4.3. In the case of \mathbb{H}^n , a Jacobi field computation gives

$$\sum_{i=1}^{kn} \left(\frac{\partial f_i}{\partial \eta} \right)^2 = \frac{1}{\sinh^2 r} \|\nabla^M r\|^2.$$

This implies that

$$\int_M \|\nabla^M \sinh r\|^2 - \int_M f^2 \sum_{i=1}^{kn} \left(\frac{\partial f_i}{\partial \eta} \right)^2 = \int_M \|\nabla^M \cosh r\|^2.$$

Thus (4.10) becomes

$$\frac{\lambda_1(M)}{\lambda_1(S(R))} \leq \frac{\text{Vol}(M)}{\text{Vol}(S(R))} + \frac{1}{(n-1)\text{Vol}(S(R))} \int_M \|\nabla^M \cosh r\|^2.$$

REFERENCES

- [1] D. Bleecker and J. Weiner.: Extrinsic bounds on λ_1 of Δ on a compact manifold. *Comment. Math. Helv.*, 51:601–609, 1976.
- [2] I. Chavel.: *Eigenvalues in Riemannian Geometry*, Academic Press, Inc. 1984.
- [3] M.P do Carmo.: *Riemannian Geometry*, Birkhäuser Boston 1992.
- [4] J. H. Eschenburg.: Comparison theorems and hypersurfaces. *Manuscripta Math.*, 59:295–323, 1987.
- [5] Grosjean J F.: Upper bounds for the first eigenvalue of the laplacian on compact submanifolds. *Pacific. J. Math.*, 206:93–112, 2002.
- [6] S. Gallot, D. Hulin and J. Lafontaine.: *Riemannian Geometry*. Third edition. Springer, 2004.
- [7] E. Heintze.: Extrinsic upperbounds for λ_1 . *Math. Ann.*, 280:389–402, 1988.
- [8] J. E. Marsden, A. J. Tromba, A. Weinstein.: *Basic Multivariable Calculus*. Springer India Pvt. Ltd, 2009.
- [9] R.Reilly.: On the first eigenvalue of the laplacian for compact submanifold of euclidean space. *Comment. Math. Helv.*, 52:525–533, 1977.
- [10] G. Santhanam.: A sharp upperbound for the first eigenvalue value of the laplacian of compact hypersurfaces in rank-1 symmetric spaces. *Proc. Indian Acad. Sci. (Math. Sci)*, 117, No. 3:307–315, August 2007.
- [11] G. Santhanam.: Isoperimetric upper bounds for the first eigenvalue. *Preprint*.
- [12] G. B. Thomas, JR and R. L. Finney.: *Calculus and Analytical Geometry*. Addison-Wesley, 9th edition.

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